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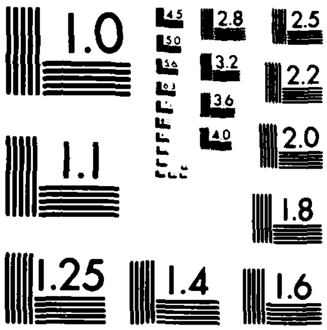

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SOME GENERALIZATIONS OF THE RENEWAL PROCESS

by

Eric Slud and Jan Winnicki
Department of Mathematics
University of Maryland
College Park, MD 20742

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SOME GENERALIZATIONS OF THE RENEWAL PROCESS

Eric Slud* and Jan Winnicki*
Department of Mathematics
University of Maryland
College Park, MD 20742

1. Introduction. The greatest success, as well as the most severe limitations, of standard Reliability Theory (as expounded, for example, in Barlow and Proschan, 1981), have been due to its restriction to the study of independent failure-time random variables. Consider the case of Renewal Theory, which in the context of Reliability has led to the characterization of many classes of repair/replacement policies, and which appears to depend crucially on the assumption of independence for the times between successive failures. In practical life, it is clear that successive replacements of failed components in a complicated assembly (say, an aircraft) may have some cumulative effect tending to shorten future times between replacements. Additionally, one can imagine that shocks to the system from failures of single components can affect the lifetimes of the remaining components, or even that the age of important components can be reflected in the operating characteristics and therefore in the hazard of failure of the system.

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The important regression models of Cox (1972) in life-table analysis gave a simple way for lifetimes to depend on (possibly time-dependent) covariate measurements. If we treat current lifetimes of system components as covariates, then these models imply an interesting and statistically estimable dependence between component failure times. This idea has been used by Slud (1982) to study a class of multivariate dependent renewal processes in which a component's hazard of failure depends only on the current component lifetimes. Another approach, which we explore in the present report, is to model the system's failure hazard as depending only on time since last failure and some cumulative exposure variable. This model and its general consequences are formulated in §2. It turns out that the most convincing generalizations of Renewal Theory are available for proportional lifetime rather than proportional hazards models, and we present them in §3. (Our general reference for Renewal Theory is Karlin and Taylor, 1975). Finally, we list in §4 some open questions and promising directions for further research.

2. <u>Formulation of models</u>. In this section we introduce a class of models generalizing the independent inter-occurrence times of renewal processes in such a way that

$$\{(T_n, Z_n)\}_{n=1}^{\infty} \quad \text{is Markovian sequence on} \quad (\Omega, F, P)$$

$$(*) \qquad Z_1 = 0, \quad Z_n = \sum_{i=1}^{n-1} T_i, \quad F_{n-1} = \sigma(T_1, T_2, \ldots, T_{n-1}) \subset F$$

$$P(T_n \geq t \mid F_{n-1}) = \bar{F}(Z_n, t) \quad \text{a.s.}$$

where $S: \mathbb{R}^+ \times \mathbb{R}^+ \to [0,1]$ is a fixed Borel-measurable function, left-continuous in its second argument. Before specializing to the important special classes of functions S, we prove some simple general results, the first of which may be slightly surprising for rapidly decreasing $S(\cdot,t)$.

Lemma 2.1. Suppose that for all $T \in (0,\infty)$ there exists $\varepsilon(T), \delta(T) > 0$ such that

$$(**) \qquad \inf\{S(z,t): 0 \le z \le T,\ 0 \le t \le \varepsilon(T)\} \ge \delta(T).$$

Then almost surely $T_n \to \infty$ as $n \to \infty$.

Proof. Assumption (*) exhibits the conditional law of T_n given F_{n-1} as a regular conditional probability. We can therefore perform the following standard construction for each T: on $(\Omega', F', P') \equiv (\Omega, F, P) \times ([0,1]^\infty, \mathcal{B}, \lambda^\infty)$ where λ^∞ denotes product Lebesgue measure, we let $\underline{u} = (u_0, u_1, \ldots) \in [0,1]^\infty$, and $U_n(\omega, \underline{u}) \equiv u_n$, so that $\{U_n\}_{n=1}^\infty$ is i.i.d. uniformly distributed and independent of $\{T_n\}_{n=1}^\infty$ (where by abuse of notation we write $T_n(\omega, \underline{u}) \equiv T_n(\omega)$): now define

$$V_n = \varepsilon(T) I_{[S(T_n, \tau_n) - U_n \beta(T_n) \le \delta(T)]}$$

where $\beta(T_n) = S(T_n, \rho(T_n)) - S(T_n, \rho(T_n)+)$ and $\rho(T_n) = \inf\{x: S(T_n, x) < \delta(T)\}$. Then it is easy to check that

$\{V_n\}_{n=1}^\infty$ is i.i.d. with $P'(V_n = \varepsilon(T)) = \delta(T) = 1 - P'(V_n = 0)$

[illegible]

$$P'(T_r \geq T_k, \text{ for } 1 \leq k \leq n, Z_n \leq T) = P'(Z_n \leq T).$$

The Strong Law of Large Numbers now implies $P'(Z_n \leq T) \to 0$ as $n \to \infty$. Hence for arbitrary $T < \infty$, $P(Z_n \leq T) \to 0$, and the lemma is proved. □

Corollary 2.2. Suppose (*) and (**) hold and also for each $t > 0$, $S(z,\cdot) \to 0$ as $z \to \infty$. Then $T_n \to 0$ in probability as $n \to \infty$.

Proposition 2.3. Assume (*) and (**). If as $z \to \infty$, $S(z,\cdot)$ converges to a left-continuous survival function $S^*(\cdot)$ with $\int_0^\infty \exp(-S^*(t))dt = \mu^* < \infty$ in the sense that

$$\inf\{\varepsilon > 0: \forall t \geq 0, S(z,t) \geq S^*(t(1+\varepsilon) \geq S(z,t(1+\varepsilon)^2)\} \to 0.$$

then as $t \to \infty$

$$N(t) \equiv \{\max n: Z_{n+1} \leq t\} \to t/\mu^* \text{ a.s.}$$

$$EN(t)/t \to 1/\mu^*$$

Proof. Fix $\varepsilon > 0$, and choose z_0 so large that for all $z \geq z_0$ and all $t \geq 0$, $S(z,t) \geq S^*(t(1+\varepsilon)) \geq S(z,t(1+\varepsilon)^2)$. For each $t > 0$, define

$$\tau(t) \equiv \inf\{Z_n: n \geq 1, Z_n \geq t\}$$

which is a.s. finite by Lemma 2.1. Conditionally on the σ-field G_t generated by the collection $\{N( ): [illegible] \tau(z_0)\}$ of random variables, the times $\{T_j: j > N(\tau(z_0))\}$ are "independently stochastically larger" than i.i.d. random variables [illegible]

with common survival curve $S^*(s(1+\varepsilon))$, in the sense that for every $m \geq 1$ and $(s_1,\ldots,s_m) \in (\mathbb{R}^+)^m$, $P\{T_{j+N(\tau(R_0))} \geq s_j$ for $j = 1,\ldots,m \mid G_0\} \geq \prod_{j=1}^{m} S^*(s_j(1+\varepsilon))$. Similarly, the random variables $\{T_j : j > N(\tau(R_0))\}$ are conditionally given G_0 jointly stochastically smaller than i.i.d. random variables $\{Y_i\}_{i=1}^{\infty}$ with common survival curve $S^*(s(1+\varepsilon)^{-1})$. It is easy to deduce that the counting-process $N(s+\tau(R_0)) - N(\tau(R_0))$ on $[0,\infty)$ is stochastically smaller in the same joint conditional sense given G_0 than the renewal counting process

$$N_X(s) \equiv \max\{k \geq 0 : \sum_{i=1}^{k} X_i \leq s\}$$

and stochastically larger than the renewal counting-process

$$N_Y(s) \equiv \max\{k \geq 0 : \sum_{i=1}^{k} Y_i \leq s\}.$$

Since the Strong Law of Large Numbers and the Basic Renewal Theorem imply for any such i.i.d. sequences $\{X_i\}_{i=1}^{\infty}$ and $\{Y_i\}_{i=1}^{\infty}$ that as $s \to \infty$

$$N_X(s)/s \to (1+\varepsilon)/\mu^*, \quad N_Y(s)/s \to (\mu^*(1+\varepsilon))^{-1} \quad \text{a.s.}$$

$$EN_X(s)/s \to (1+\varepsilon)/\mu^*, \quad EN_Y(s) \to (\mu^*(1+\varepsilon))^{-1}.$$

We conclude (by a known construction similar to but more complicated than that used in proving Lemma 2.1, which would define processes $N_Y(\cdot) \leq N(\cdot+\tau(R_0)) - N(\tau(R_0)) \leq N_X(\cdot)$ on the same probability space) that

$$\text{a.s.} \limsup_{s\to\infty} s^{-1}[N(s+\tau(R_0)) - N(\tau(R_0))] \le (1+\epsilon)/\mu^*$$

$$\text{a.s.} \limsup_{s\to\infty} s^{-1}[N(s+\tau(R_0)) - N(\tau(R_0))] \ge (\mu^*(1+\epsilon))^{-1}$$

with similar statements for expectations. Since ϵ is arbitrary and the sequences of random variables $s^{-1}(N(s+\tau(R_0)) - N(s))$ and $s^{-1}(N(\tau(R_0))$ are uniformly integrable for $s \ge 1$, say, with expectations converging to 0, and since obviously

$$s^{-1}N(\tau(R_0)) \to 0 \text{ a.s. as } s \to \infty$$

the Proposition is proved. □

The functions $S(\cdot,\cdot)$ of greatest interest to us will be those which are non-increasing in their first arguments. With or without this extra assumption, we restrict further attention to the following subclasses of examples.

1° *Proportional hazard models.* If $S(z,t) \equiv \exp[-Q(z)\Lambda(t)]$, we see that the interoccurrence time T_n has conditional cumulative hazard $Q(Z_n)\Lambda(\cdot)$ given F_{n-1} with the factor $Q(Z_n)$ multiplying the hazard function $\Lambda(\cdot)$ of T_1 (where we assume $Q(0) = 1$). Such models were first introduced into failure-time analysis by Cox (1972).

2° *Proportional time models.* If $S(z,t) \equiv S_0(t/q(z))$, where $q(0) = 1$ and $S_0(\cdot)$ is the left-continuous survival function for T_1, then (*) implies that the random variables $T_n/q(Z_n) \equiv W_n$ are independent and identically distributed with common survival function $S_0(\cdot)$.

In case $S_0(\cdot)$ has the Weibull form $\exp(-\lambda t^{\gamma})$, then models 1° and 2° are completely equivalent, as is well known. In model 1°, (**) says simply that $\sup\{Q(z): 0 \leq z \leq T\} < \infty$ for each $T < \infty$. In model 2°, (**) becomes: $\inf\{q(z): 0 \leq z \leq T\} > 0$ for $T < \infty$. Proposition 2.3 applies directly to model 2° whenever (*) and (**) hold and $q(z) \to q_* \in (0,\infty)$ as $z \to \infty$, and then its additional hypothesis with $S^*(t) = S_0(t/q_*)$ follows. However, the Proposition applies to model 1° only for very special $\Lambda(\cdot)$.

3. Renewal theory for proportional time models.

Now we assume (*) with $S(z,t) \equiv S_0(t/q(z))$ as in 2° above, and let $\{W_n\}_{n=1}^{\infty}$ be the i.i.d. sequence given by $W_n = T_n/q(Z_n)$, so that

$$Z_{n+1} = \sum_{j=1}^{n} q(Z_j)W_j. \tag{3.1}$$

Throughout the present section, we assume that $q(\cdot)$ is non-increasing. In what follows, we require the definitions

$$N(t) \equiv \max\{k: Z_{k+1} \leq t\}$$

$$\tau(t) \equiv \inf\{Z_k: k \geq 1, Z_k \geq t\}$$

as well as the observation that the random variables $\tau(t)$ are stopping times with respect to the family of σ-fields

$$F^t \quad \sigma(\{N(s): 0 \leq s \leq t\}).$$

Let $0 < u < t$ be arbitrary. By the definitions and Wald's identity,

$$E(\tau(t)-\tau(u)) = E\left(\sum_{j=N(\tau(u))+1}^{N(\tau(t))} q(Z_j)W_j\right)$$

$$= E\left(\sum_{j=N(\tau(u))+1}^{N(\tau(t))} q(Z_j)\mu\right)$$

where we have assumed $\mu = EW_i = ET_1 < \infty$. However, for $j \leq N(\tau(t))$, $q(Z_j) \geq q(t)$, so that we have proved

Lemma 3.1. If (*) and (**) hold, $q(\cdot)$ is nonincreasing, $\mu = ET_1 < \infty$, and $0 < u < t$, then

$$E(N(t)) - N(u)) = E(N(\tau(t)) - N(\tau(u)))$$

$$\leq E(\tau(t) - \tau(u))/(\mu q(t)).$$

Our next lemma depends upon an idea already used in the proof of Proposition 2.3, namely that a stochastic order relation between lifetimes T_j and i.i.d. lifetimes W_j leads to a stochastic order relation between $N(\cdot)$ and the renewal counting process associated with $\{W_j\}$.

Lemma 3.2. Under the same hypotheses as in Lemma 3.1,

$$E(N(t) - N(u)) \leq 1 + EN_W\left(\frac{t-u}{q(t)}\right)$$

where

$$N_W(x) \equiv \max\{k: \sum_{j=1}^{k} W_j \leq x\}.$$

Proof. As before, $N(t) - N(u) = 1 + N(t) - N(\tau(u))$ a.s. Moreover, for $N(\tau(u)) + 1 \leq j \leq N(\tau(t))$, conditionally given $\sigma(\tau(u), \{N(s): s \leq \tau(u)\})$ the lifetimes T_j are [illegible]

stochastically larger than the i.i.d. random variables [illegible].
So that the process $N(\tau(u)+\cdot) - N(\tau(u))$ is conditionally sto-
chastically smaller than $N_W(\cdot/q(t))$. Therefore

a.s. $$E(N(t) - N(\tau(u)) \mid \sigma(\tau(u), \{N(s) : s \leq \tau(u)\}) \leq$$

$$EN_W((t - \tau(u))/q(t)) \leq EN_W((t-u)/q(t)).$$

Therefore $E(N(t) - N(u)) = 1 + E(N(t) - N(\mu(u))) \leq$
$EN_W((t-u)/q(t)) + 1$. □

Lemma 3.3. Suppose that $q(\cdot)$ is a non-increasing Borel measurable function on $[0,\infty)$ with $\int_0^\infty q(x)dx < \infty$ but $q(x) > 0$ for all $x < \infty$. Then there exists a real sequence $\{t_j\}_{j=0}^\infty$ increasing to ∞ such that $t_0 = 0$ and

(a) $\sum_{j=1}^{\infty} q(t_j)(t_{j+1} - t_j) < \infty$

(b) $(t_{j+1} - t_j)/q(t_j) \to \infty$ as $j \to \infty$.

Proof. Fix any constant $K > 0$, and define $\{s_j\}_{j=0}^\infty$ by $s_0 = 0$, $s_{j+1} - s_j = Kq(s_j)$. Then $s_j \nearrow \infty$ as $j \to \infty$; for if $s_j \nearrow s^* < \infty$ then $s_{j+1} \nearrow s^* + K \lim_{j\to\infty} q(s_j) > s^*$, which is a contradiction. The properties of $q(\cdot)$ now imply

$$\sum_{j=0}^{\infty} q(s_{j+1})(s_{j+1}-s_j) \leq \int_0^\infty q(x)dx < \infty.$$

Therefore, by definition of $s_{j+1}-s_j$, $\sum_{j=0}^{\infty} q(s_{j+1})Kq(s_j) < \infty$.

Hence $\sum_{j=1}^{\infty} q^2(s_j) < \infty$, and there exists a real sequence $\{\alpha_j\}$ increasing to ∞ slowly enough so that $\sum_{j=1}^{\infty} \alpha_j q^2(s_j) < \infty$.

Now define $\{t_j\}$ by $t_0 = 0$ and $t_{j+1} - t_j = \alpha_j q(t_j) + Kq(s_j)$. Clearly $t_j \geq s_j$ for all $j \geq 0$, so that

$$\sum_{j=1}^{\infty} q(t_j)(t_{j+1} - t_j) = \sum_{j=1}^{\infty} [\alpha_j q^2(t_j) + Kq(t_j)q(s_j)] \leq \sum_{j=1}^{\infty} (\alpha_j + K) q^2(s_j) < \infty .$$

The lemmas now allow us to prove a striking generalization of the Basic Renewal Theorem to proportional time models with nonincreasing integrable $q(\cdot)$.

Theorem 3.4. Suppose that (*) holds for $S(z,t) = S_0(t/q(z))$, where $q(\cdot)$ is a nonincreasing strictly positive Lebesgue integrable function on $[0,\infty)$ with $q(0) = 1$. Suppose also that $\mu = ET_1$ and $\sigma^2 = E(T_1 - \mu)^2$ is finite. (Since $P(T_1 \geq t) = S_0(t)$, this is equivalent to assuming $\int_0^{\infty} tS_0(t)dt < \infty$.) Then

(i) $E \sum_{j=1}^{\infty} q^2(Z_j) < \infty$

(ii) a.s. $\lim_{n\to\infty} (Z_n - \mu \sum_{j=1}^{n-1} q(Z_j)) \equiv \Delta < \infty$ exists

(iii) $Z_n \to \infty$ almost surely, as $n \to \infty$

(iv) If $q(\cdot)$ is continuous, then $\frac{\mu}{t} \int_0^t q(s) dN(s) \to 1$ as $t \to \infty$, almost surely and in the mean.

<u>Proof</u>. (i) Fix the sequence $\{t_j\}$ constructed in Lemma [illegible]. Then (again using Wald's identity)

$$E \sum_{j=N(t_i)+2}^{N(t_{i+1})+1} q^2(Z_j) \le q(t_i)\mu^{-1} E\left\{ \sum_{j=N(t_i)+2}^{N(t_{i+1})} q(Z_j)W_j + q(t_i)\mu \right\}.$$

By the representation (3.1), the right-hand side is $\le q^2(t_i) + q(t_i)\mu^{-1}(t_{i+1}-t_i)$. Therefore

$$E \sum_{j=1}^{\infty} q^2(Z_j) \le 1 + \sum_{i=1}^{\infty} [q^2(t_i) + q(t_i)\mu^{-1}(t_{i+1}-t_i)] < \infty$$

where finiteness of the sum follows from Lemma 3.3.

(ii) The sequence $Z_n - \sum_{j=1}^{n-1} \mu q(Z_j)$ is obviously a F_{n-1} adapted martingale with

$$\sup_{n \ge 1} E\{Z_n - \sum_{j=1}^{n-1} \mu q(Z_j)\}^2 = \sup_{n \ge 1} E\left\{ \sum_{j=1}^{n-1} q(Z_j)(W_j - \mu) \right\}^2$$

$$= \sup_{n \ge 1} E \sum_{j=1}^{n-1} q^2(Z_j)\sigma^2 = \sigma^2 E \sum_{j=1}^{\infty} q^2(Z_j) < \infty.$$

Thus $\{Z_n - \mu \sum_{j=1}^{n-1} q(Z_j)\}$ is a square-integrable, hence uniformly integrable, martingale sequence, which by the Martingale Convergence Theorem has a finite almost-sure limiting random variable Δ.

(iii) It follows from (ii) that a.s. as $n \to \infty$, $Z_{n+1} - Z_n - \mu q(Z_n) \to 0$. Now $Z_{n+1} - Z_n = T_n$, while $q(\cdot)$ integrable non-increasing implies $q(Z_n) \to 0$ a.s. since $Z_n \to \infty$ (Lemma 3.1). Therefore $T_n \to 0$ a.s.

(iv) Since $T_n \to 0$ a.s., as $t \to \infty$, $q(\cdot) \to 0$ almost surely. Now fix arbitrarily small $\varepsilon > 0$ and let $\{r_j\}$ be an increasing sequence such that $q(r_{j+1}) = (1+\varepsilon)^{-1} q(r_j)$ for all j. Then if we define k by $r_k < t \leq r_{k+1}$, we find by (ii) for $j < k$ with $r_j, t \to \infty$

$$\tau(t)-\tau(r_j) = \mu \sum_{z=j+1}^{k} \vartheta_i(N(r_i)-N(r_{i-1})) + \vartheta_{k+1}(N(t)-N(r_k)) + \zeta_{j,t}$$

where $q(r_{i-1}) \geq \vartheta_i \geq q(r_i)$ a.s. and $\zeta_{j,t} \to 0$ as $t, j \to \infty$. Therefore as $r_j, t \to \infty$, $r_j < t$,

$$\tau(t) - \tau(r_j) \leq \mu(1+\varepsilon) \int_{r_j}^{t} q(s)dN(s) + \zeta_{j,t}$$

$$\tau(t) - \tau(r_j) \geq \mu(1+\varepsilon)^{-1} \int_{r_j}^{t} q(s)dN(s) + \zeta_{j,t}.$$

Since $\tau(t) - t \to 0$ a.s. and in the mean; since r_j may be arbitrarily much smaller than t; since $\varepsilon > 0$ is arbitrary, we conclude a.s. $t \to \infty$

$$\frac{\mu}{t} \int_0^t q(s)dN(s) \to 1 \text{ a.s. and in } L^1. \quad \Box$$

Theorem 3.4(iv) is a direct extension of the Basic Renewal Theorem (which gives the same statement when $q \equiv 1$). Various asymptotic forms for $N(t)$, Z_n and $EN(t)$ can be derived from the result and proof of Theorem 3.4(iv) under further conditions on the rate of decrease of $q(\cdot)$. The typical statement (cf. [illegible]) is easy to establish for regularly varying $q(\cdot)$ but [illegible]

much more generally) is

$$(3.2)\qquad \begin{aligned} N(t) &\sim \mu^{-1}\int_0^t ds/q(s) \equiv R(t)/\mu && \text{a.s. as } t \to \infty \\ Z_n &\sim R\mu^{-1}(n) && \text{a.s. as } n \to \infty. \end{aligned}$$

4. Open problems. Directions for further research. There are of course many technical improvements possible in the foregoing results. We list instead some of the more important questions related to our generalization of renewal theory which our techniques are so far completly unable to answer.

(A) In the models *1* and *2*, under the hypotheses of Proposition 2.3, is there any reasonably general asymptotic expansion for $EN(t) - t/\mu^*$? Does the Renewal Theorem itself have a natural generalization?

(B) Do any of the results of Section 3 have analogues for the proportional hazards models?

(C) Does Theorem 3.4 hold with the hypothesis of integrability of $q(\cdot)$ weakened to: $q(z) \to 0$ as $z \to \infty$?

(D) In cases of very mild decrease for $q(\cdot)$ or increase for $Q(\cdot)$ in models *1*, *2*, are there any practical methods of calculating or approximating $EN(t)$ for small or moderate t?

The most interesting variants of the models we have introduced, and which will be treated in a future report would retain (*) in its entirety except for the modified definition

$$Z_n \equiv \sum_{j=1}^{n} w(T_j)$$

where the (nondecreasing) function $w(\cdot)$ [illegible] additive damage of the system [illegible] between successive failures. The results of [illegible], [illegible] should have modifications holding for this variant model. Further variant might allow a number of such variables [illegible] affect the law of T_n.

The functions $q(\cdot)$ and $g(\cdot)$ [illegible] this report, and $w(\cdot)$ in the previous paragraph, may for potential applications be supposed to depend on (possibly unknown) parameters and [illegible] random variables, in which case statistical procedures for estimating unknown parameters will be of interest. This [illegible] subject for future research.

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